ON THE PROBLEM OF THE ELASTIC EQUILIBRIUM OF AN ANISOTROPIC STRIP

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The plane problem on the distribution of stresses in an elastic, anisotropic strip, deformed by loadings of particular and general form, has been treated by various authors. In [1-5] the solutions are constructed with the aid of Airy's stress function in the form of polynomial or trigonometric series. Kufarev and Sveklo [6] and Shepelenko [7] have made use of the complex representation of the stresses and displacements; the solutions of the two basic boundary value problems, and of some mixed ones, were found by means of Fourier integrals.

Below, an account will be given of the general operator method of solution of the present problem, which is analogous to the method of A.I. Lur'e with the aid of which elegant solutions of simple form have been obtained for three-variable problems on the elastic equilibrium of the isotropic layer and of the thick plate ([8, Chaps. 3 and 4] and [9]).

1. Formulation of the problem and the general equations. We consider an infinite, elastic, anisotropic strip of constant with h, which is in a state of generalized plane stress or plane strain under the action of a self-equilibrating system of forces distributed along the edges. We will assume that the material obeys the generalized Hooke's law and that the strains are small. The x-axis is directed along the axis of the strip and the y-axis is normal to the boundary.

When there is only one plane of elastic symmetry, parallel to the xy plane, the equations of generalized Hooke's law can be written down in the following form

$$\begin{aligned} \varepsilon_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy}, & \gamma_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy} \\ \varepsilon_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy}, & \gamma_{yz} &= a_{44}\tau_{yz} + a_{45}\tau_{xz} \\ \varepsilon_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy}, & \gamma_{xz} &= a_{45}\tau_{yz} + a_{55}\tau_{xz} \end{aligned} \tag{1.1}$$

where a_{ij} are the strain coefficients; they can be expressed in terms of Young's modulus, the shear modulus, and Poisson's ratio, etc. (see, e.g. [1, p.18]).

Henceforth we will denote the derivative with respect to x by ∂ ($\partial = \partial/\partial x$) and the derivative with respect to y by a dash. Then the formulas for the stresses in the plane problem and the equation for the stress function can be written down as

$$\sigma_x = F'', \qquad \sigma_y = \partial^2 F, \qquad \tau_{xy} = -\,\partial F' \tag{1.2}$$

$$\beta_{11}F^{\mathrm{IV}} - 2\beta_{16}\partial F''' + (2\beta_{12} + \beta_{66})\,\partial^2 F'' - 2\beta_{26}\partial^3 F' + \beta_{22}\partial^4 F = 0 \tag{1.3}$$

In the case of generalized plane stress, the constants $\beta_{ij} = a_{ij}$, and in the case of plane strain $\beta_{ij} = a_{ij} - a_{i3}a_{j3} : a_{33}$.

We will seek the solution of equation (1.3) in the form of the series

$$F = \sum_{k=0}^{\infty} g_k(x)\, y^k \tag{1.4}$$

Substituting this expression into equation (1.3) and equating to zero the coefficients of each power of y, we obtain an infinite system of recurrence equations connecting the g with different indices and their derivatives up to and including the fourth. This system allows one to express all functions in terms of four, e.g. g_0, g_1, g_2, g_3. The characteristic equation, corresponding to (1.3), has the form

$$\beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\,\mu^2 - 2\beta_{26}\mu + \beta_{22} = 0 \tag{1.5}$$

the roots of which, called the complex parameters of the plane problem, will be denoted by μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$. When the complex parameters are distinct the general expression for F can be written down in the following complex form

$$F = e^{\mu_1 y\partial}\cdot\varphi_1 + e^{\mu_2 y\partial}\cdot\varphi_2 + e^{\bar{\mu}_1 y\partial}\cdot\bar{\varphi}_1 + e^{\bar{\mu}_2 y\partial}\cdot\bar{\varphi}_2 \tag{1.6}$$

where

$$e^{\mu y\partial} = 1 + \mu y\partial + \frac{\mu^2 y^2}{2!}\,\partial^2 + \frac{\mu^3 y^3}{3!}\,\partial^3 + \ldots \tag{1.7}$$

The arbitrary functions φ_1, φ_2 of the variable x (generally speaking, with complex coefficients) and the conjugate functions $\bar{\varphi}_1$, $\bar{\varphi}_2$ can be determined from the boundary conditions on the sides of the strip $y = \pm h/2$.

The points in (1.6) and the following formulas will be used to separate the differential operators from the function to which they are applied.

For simplifying the computations in the sequel we will only consider an orthotropic strip. In this case we have $a_{16} = a_{26} = a_{36} = a_{45} = 0$ in equations (1.1), and consequently $\beta_{16} = \beta_{26} = 0$. The displacement components u, v can be determined in terms of the known stresses by the equations

$$\partial u = \beta_{11}\sigma_x + \beta_{12}\sigma_y, \qquad v' = \beta_{12}\sigma_x + \beta_{22}\sigma_y, \qquad \partial v + u' = \beta_{66}\tau_{xy} \tag{1.8}$$

We introduce the new parameters $\pm s_1$, $\pm s_2$ which are the roots of the equation

$$\beta_{11}s^4 - (2\beta_{12} + \beta_{66})\, s^2 + \beta_{22} = 0 \tag{1.9}$$

When β_{11} and β_{22} are finite and non-zero the following three cases for the roots are possible

$$(1)\ s_1 = \beta,\ s_2 = \delta; \qquad (2)\ s_1 = \beta,\ s_2 = \beta; \qquad (3)\ s_1 = \beta + \alpha i,\ s_2 = \beta - \alpha i$$

(α, β, δ are positive, real numbers).

If $s_1 \neq s_2$ the general expression for the stress function can be written down as

$$F = \cos s_1\, y\partial.\ F_1 + \cos s_2\, y\partial.\ F_2 + \sin s_1\, y\partial.\ F_1^* + \sin s_2\, y\partial.\ F_2^* \tag{1.10}$$

Introducing the new notation for the unknown functions

$$f_k = \partial F_k, \qquad f_k^* = \partial F_k^* \tag{1.11}$$

we obtain by formula (1.2) the expressions for the stresses

$$\begin{gathered}\sigma_x = -\partial\,(s_1^2\cos s_1\, y\partial.\ f_1 + s_2^2\cos s_2\, y\partial.\ f_2 + s_1^2\sin s_1\, y\partial.\ f_1^* + s_2^2\sin s_2\, y\partial.\ f_2^*)\\ \sigma_y = \partial\,(\cos s_1\, y\partial.\ f_1 + \cos s_2\, y\partial.\ f_2 + \sin s_1\, y\partial.\ f_1^* + \sin s_2\, y\partial.\ f_2^*)\\ \tau_{xy} = \partial\,(s_1\sin s_1\, y\partial.\ f_1 + s_2\sin s_2\, y\partial.\ f_2 - s_1\cos s_1\, y\partial.\ f_1^* - s_2\cos s_2\, y\partial.\ f_2^*)\end{gathered} \tag{1.12}$$

Integrating the equations (1.8) we find the general formulas for the displacements

$$\begin{gathered}u = (\beta_{12} - \beta_{11}s_1^2)(\cos s_1\, y\partial.\ f_1 + \sin s_1\, y\partial.\ f_1^*) +\\ + (\beta_{12} - \beta_{11}s_2^2)(\cos s_2\, y\partial.\ f_2 + \sin s_2\, y\partial.\ f_2^*) - \omega y + u_0\end{gathered} \tag{1.13}$$

$$v=\left(\frac{\beta_{22}}{s_1}-\beta_{12}s_1\right)(\sin s_1\, y\partial.\ f_1-\cos s_1\, y\partial.\ f_1{}^*)+$$
$$+\left(\frac{\beta_{22}}{s_2}-\beta_{12}s_2\right)(\sin s_2.\ y\partial\ f_2-\cos s_2\, y\partial.\ f_2{}^*)+\omega x+v_0$$

(ω, u_0, v_0 are arbitrary constants characterizing "rigid" displacements in the x, y plane).

In order not to complicate the problem too much we will assume that in the whole region on each of two straight edges of the strip either the stresses or the displacements, or one of the stress components and one of the components of the displacement, are prescribed. Satisfying the boundary conditions, we will obtain a system of four differential equations (of infinitely high order) for the four unknown functions of the variable x and the problem has been reduced to the integration of this system. For the orthotropic strip it is meaningful to resolve the prescribed tractions or displacements into components symmetric and antisymmetric with respect to the x-axis and to find the corresponding distributions of stress, i.e. the symmetric and the antisymmetric solutions. We will limit consideration to the case of prescribed tractions; the solution of the second basic problem and of the mixed ones can be found in an identical manner.

If the strip has finite length, the conditions on the end faces - in the present work - will not be satisfied exactly and we will limit ourselves to satisfying only the condition that the resultant force and resultant moment of the loading agree with those of the prescribed loading.

However, we note that the operator method makes possible the derivation of more exact solutions.

2. Symmetric distribution of tractions. Let the two sides of the strip be subjected to the tractions p, τ (per unit area) which are symmetric with respect to the x-axis (Fig. 1). We have the boundary conditions

$$\sigma_y=p(x),\qquad \tau_{xy}=\pm\tau(x)\quad \text{when } y=\pm h/2 \tag{2.1}$$

In an orthotropic strip the distribution of stresses will also be symmetric and it is possible to take $f_1{}^*=f_2{}^*=0$ in advance in formulas (1.12) and (1.13). Assuming that $s_1\neq s_2$, the boundary conditions lead to a system of two equations for f_1 and f_2

$$\partial\left(\cos\frac{s_1h\partial}{2}\cdot f_1+\cos\frac{s_2h\partial}{2}\cdot f_2\right)=p$$

$$\partial\left(s_1 \sin \frac{s_1 h\partial}{2}\cdot f_1 + s_2 \sin \frac{s_2 h\partial}{2}\cdot f_2\right) = \tau \tag{2.2}$$

We denote the differential operator proportional to the determinant of the system of equation (2.2) by Q

$$\begin{aligned} Q &= \frac{1}{s_1 - s_2}\left(s_1 \sin \frac{s_1 h\partial}{2} \cos \frac{s_2 h\partial}{2} - s_2 \sin \frac{s_2 h\partial}{2} \cos \frac{s_1 h\partial}{2}\right) = \\ &= 0.5\left[\frac{s_1 + s_2}{s_1 - s_2} \sin (s_1 - s_2)\ \frac{h\partial}{2} + \sin (s_1 + s_2) \frac{h\partial}{2}\right] \end{aligned} \tag{2.3}$$

We introduce the stress functions χ_1 and χ_2 satisfying the equations

$$Q(\partial\chi_1) = p, \qquad Q(\partial\chi_2) = \tau \tag{2.4}$$

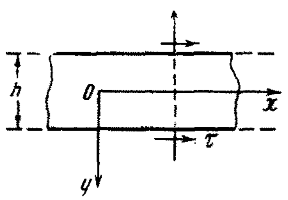

Fig. 1.

For this it suffices to set

$$\begin{aligned} f_1 &= \frac{1}{s_1 - s_2}\left(-s_2 \sin \frac{s_2 h\partial}{2}\cdot\chi_1 + \cos \frac{s_2 h\partial}{2}\cdot\chi_2\right) \\ f_2 &= \frac{1}{s_1 - s_2}\left(s_1 \sin \frac{s_1 h\partial}{2}\cdot\chi_1 - \cos \frac{s_1 h\partial}{2}\cdot\chi_2\right) \end{aligned} \tag{2.5}$$

Substituting these values in (1.12), we obtain the formulas for the stresses

$$\begin{aligned} \sigma_x &= \frac{s_1 s_2}{s_1 - s_2}\left(s_1 \sin \frac{s_2 h\partial}{2} \cos s_1 y\partial - s_2 \sin \frac{s_1 h\partial}{2} \cos s_2 y\partial\right)\partial\chi_1 - \\ &\quad - \frac{1}{s_1 - s_2}\left(s_1^2 \cos \frac{s_2 h\partial}{2} \cos s_1 y\partial - s_2^2 \cos \frac{s_1 h\partial}{2} \cos s_2 y\partial\right)\partial\chi_2 \\ \sigma_y &= -\frac{1}{s_1 - s_2}\left(s_2 \sin \frac{s_2 h\partial}{2} \cos s_1 y\partial - s_1 \sin \frac{s_1 h\partial}{2} \cos s_2 y\partial\right)\partial\chi_1 + \\ &\quad + \frac{1}{s_1 - s_2}\left(\cos \frac{s_2 h\partial}{2} \cos s_1 y\partial - \cos \frac{s_1 h\partial}{2} \cos s_2 y\partial\right)\partial\chi_2 \\ \tau_{xy} &= -\frac{s_1 s_2}{s_1 - s_2}\left(\sin \frac{s_2 h\partial}{2} \sin s_1 y\partial - \sin \frac{s_1 h\partial}{2} \sin s_2 y\partial\right)\partial\chi_1 + \\ &\quad + \frac{1}{s_1 - s_2}\left(s_1 \cos \frac{s_2 h\partial}{2} \sin s_1 y\partial - s_2 \cos \frac{s_1 h\partial}{2} \sin s_2 y\partial\right)\partial\chi_2 \end{aligned} \tag{2.6}$$

The expressions for the displacements will not be quoted because their structure is clear from (1.13) and (2.5). In the formulas for the displacements, not the first derivatives of χ_1 and χ_2, but the functions themselves appear.

In case (3), when $s_1 = \beta + \alpha i$, $s_2 = \beta - \alpha i$, in place of (2.3) we will have

$$Q = 0.5\left(\frac{\beta}{\alpha}\sinh \alpha h\partial + \sin \beta h\partial\right) \tag{2.7}$$

When $s_1 = s_2 = \beta$

$$Q = 0.5\,(\beta h\partial + \sin \beta h\partial) \tag{2.8}$$

and the expressions for the stresses are obtained from (2.6) by letting $\alpha \to 0$

$$\begin{aligned}
\sigma_x = {} & \beta^2 \left[\left(\sin\frac{\beta h\partial}{2} - \frac{\beta h\partial}{2}\cos\frac{\beta h\partial}{2}\right)\cos\beta y\partial - \sin\frac{\beta h\partial}{2}\,\beta y\partial\sin\beta y\partial\right]\partial\chi_1 - \\
& - \beta\left[\left(2\cos\frac{\beta h\partial}{2} + \frac{\beta h\partial}{2}\sin\frac{\beta h\partial}{2}\right)\cos\beta y\partial - \cos\frac{\beta h\partial}{2}\,\beta y\partial\sin\beta y\partial\right]\partial\chi_2 \\
\sigma_y = {} & \left[\left(\sin\frac{\beta h\partial}{2} + \frac{\beta h\partial}{2}\cos\frac{\beta h\partial}{2}\right)\cos\beta y\partial + \sin\frac{\beta h\partial}{2}\,\beta y\partial\sin\beta y\partial\right]\partial\chi_1 + \\
& + \left(\frac{h}{2}\sin\frac{\beta h\partial}{2}\cos\beta y\partial - \cos\frac{\beta h\partial}{2}\,y\sin\beta y\partial\right)\partial^2\chi_2 \\
\tau_{xy} = {} & -\beta^2\left(\sin\frac{\beta h\partial}{2}\,y\cos\beta y\partial - \frac{h}{2}\cos\frac{\beta h\partial}{2}\sin\beta y\partial\right)\partial^2\chi_1 + \\
& + \left[\left(\cos\frac{\beta h\partial}{2} + \frac{\beta h\partial}{2}\sin\frac{\beta h\partial}{2}\right)\sin\beta y\partial + \cos\frac{\beta h\partial}{2}\,\beta y\partial\cos\beta y\partial\right]\partial\chi_2
\end{aligned} \tag{2.9}$$

Then, when $\beta = 1$, we obtain the stress distribution in an isotropic strip ($s_1 = s_2 = 1$). By means of integration across the width of the strip it is easy to veryfy that the stresses in every cross-section reduce to an axial force depending only on χ_2, i.e. on the given shear tractions

3. Antisymmetric distribution of tractions. For an antisymmetric distribution of external loading (Fig. 2), the following boundary conditions must be satisfied

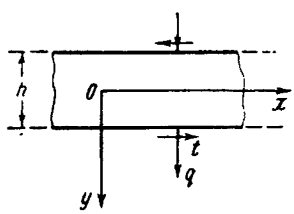

Fig. 2.

$$\sigma_y = \pm q\,(x), \qquad \tau_{xy} = t\,(x) \quad \text{for } y = \pm\frac{h}{2} \tag{3.1}$$

Assuming $f_1 = f_2 = 0$, we obtain for f_1^*, f_2^* the system (when $s_1 \neq s_2$)

$$\begin{aligned}
&\partial\left(\sin\frac{s_1 h\partial}{2}\cdot f_1^* + \sin\frac{s_2 h\partial}{2}\cdot f_2^*\right) = q \\
&\partial\left(s_1\cos\frac{s_1 h\partial}{2}\cdot f_1^* + s_2\cos\frac{s_2 h\partial}{2}\cdot f_2^*\right) = -\,t
\end{aligned} \tag{3.2}$$

We introduce the stress functions ψ_1, ψ_2 such that

$$f_1^* = s_2\cos\frac{s_2 h\partial}{2}\cdot\psi_1 + \sin\frac{s_2 h\partial}{2}\cdot\psi_2, \qquad f_2^* = -\,s_1\cos\frac{s_1 h\partial}{2}\cdot\psi_1 - \sin\frac{s_1 h\partial}{2}\cdot\psi_2 \tag{3.3}$$

and we obtain for them the equations

$$Q^*(\partial\psi_1) = q, \qquad Q^*(\partial\psi_2) = t \tag{3.4}$$

where

$$\begin{aligned} Q^* &= \frac{1}{s_1 - s_2}\left(s_2 \sin\frac{s_1 h\partial}{2}\cos\frac{s_2 h\partial}{2} - s_1 \sin\frac{s_2 h\partial}{2}\cos\frac{s_1 h\partial}{2}\right) = \\ &= 0.5\left[\frac{s_1 + s_2}{s_1 - s_2}\sin(s_1 - s_2)\frac{h\partial}{2} - \sin(s_1 + s_2)\frac{h\partial}{2}\right] \end{aligned} \tag{3.5}$$

The stresses can be determined by the formulas

$$\begin{aligned} \sigma_x &= -\frac{s_1 s_2}{s_1 - s_2}\left(s_1 \cos\frac{s_2 h\partial}{2}\sin s_1 y\partial - s_2 \cos\frac{s_1 h\partial}{2}\sin s_2 y\partial\right)\partial\psi_1 - \\ &\quad - \frac{1}{s_1 - s_2}\left(s_1^2 \sin\frac{s_2 h\partial}{2}\sin s_1 y\partial - s_2^2 \sin\frac{s_1 h\partial}{2}\sin s_2 y\partial\right)\partial\psi_2 \\ \sigma_y &= \frac{1}{s_1 - s_2}\left(s_2 \cos\frac{s_2 h\partial}{2}\sin s_1 y\partial - s_1 \cos\frac{s_1 h\partial}{2}\sin s_2 y\partial\right)\partial\psi_1 + \\ &\quad + \frac{1}{s_1 - s_2}\left(\sin\frac{s_2 h\partial}{2}\sin s_1 y\partial - \sin\frac{s_1 h\partial}{2}\sin s_2 y\partial\right)\partial\psi_2 \\ \tau_{xy} &= -\frac{s_1 s_2}{s_1 - s_2}\left(\cos\frac{s_2 h\partial}{2}\cos s_1 y\partial - \cos\frac{s_1 h\partial}{2}\cos s_2 y\partial\right)\partial\psi_1 - \\ &\quad - \frac{1}{s_1 - s_2}\left(s_1 \sin\frac{s_2 h\partial}{2}\cos s_1 y\partial - s_2 \sin\frac{s_1 h\partial}{2}\cos s_2 y\partial\right)\partial\varphi_2 \end{aligned} \tag{3.6}$$

In case (3)

$$Q^* = 0.5\left(\frac{\beta}{\alpha}\sinh\alpha h\partial - \sin\beta h\partial\right) \tag{3.7}$$

When $s_1 = s_2 = \beta$

$$Q^* = 0.5(\beta h\partial - \sin\beta h\partial) \tag{3.8}$$

$$\begin{aligned} \sigma_x &= -\beta^2\left[\left(\frac{\beta h\partial}{2}\sin\frac{\beta h\partial}{2} + \cos\frac{\beta h\partial}{2}\right)\sin\beta y\partial + \cos\frac{\beta h\partial}{2}\beta y\partial\cos\beta y\partial\right]\partial\psi_1 - \\ &\quad - \beta\left[\left(2\sin\frac{\beta h\partial}{2} - \frac{\beta h\partial}{2}\cos\frac{\beta h\partial}{2}\right)\sin\beta y\partial + \sin\frac{\beta h\partial}{2}\beta y\partial\cos\beta y\partial\right]\partial\psi_2 \\ \sigma_y &= \left[\left(\frac{\beta h\partial}{2}\sin\frac{\beta h\partial}{2} - \cos\frac{\beta h\partial}{2}\right)\sin\beta y\partial + \cos\frac{\beta h\partial}{2}\beta y\partial\sin\beta y\partial\right]\partial\psi_1 - \\ &\quad + \left(\sin\frac{\beta h\partial}{2}y\cos\beta y\partial - \frac{h}{2}\cos\frac{\beta h\partial}{2}\sin\beta y\partial\right)\partial^2\psi_2 \\ \tau_{xy} &= \beta^2\left(\cos\frac{\beta h\partial}{2}y\sin\beta y\partial - \frac{h}{2}\sin\frac{\beta h\partial}{2}\cos\beta y\partial\right)\partial^2\psi_1 + \\ &\quad + \left[\left(\frac{\beta h\partial}{2}\cos\frac{\beta h\partial}{2} - \sin\frac{\beta h\partial}{2}\right)\cos\beta y\partial + \sin\frac{\beta h\partial}{2}\beta y\partial\sin\beta y\partial\right]\partial\psi_2 \end{aligned} \tag{3.9}$$

Across a cross-section the stresses (3.6) and (3.9) can be reduced, in general, to a bending moment and a shearing force acting in the plane of the cross-section.

We note that the method of solution of the problem for a non-orthotropic strip turns out to be essentially the same as that for an orthotropic strip, and it is only necessary to proceed from the more complicated expression for the stress function (1.6). The boundary conditions give rise to a system of four equations for the functions φ_1, φ_2, $\bar{\varphi}_1$, $\bar{\varphi}_2$, which in the general case cannot be resolved into two systems similar to (2.2) and (3.2).

4. Stresses in a strip of finite length. In the case of a strip of finite length l it is necessary to satisfy conditions not only on the sides $y = \pm h/2$ but also on the end faces. For this, use must be made of so-called homogeneous solutions which correspond to the case when the edges $y = \pm h/2$ are free of traction ($p = \tau = 0$, $q = t = 0$). Each of the equations

$$Q(\partial\chi_k) = 0, \qquad Q^*(\partial\psi_k) = 0 \qquad (k = 1,2) \tag{4.1}$$

has an infinity of solutions, but we only consider the simplest homogeneous solutions which can be obtained by retaining only lower powers of ∂ in the expansions of the operators.

A) *Symmetric distribution*. Discarding in the expression Q powers of ∂ higher than the first, we obtain the equations

$$\partial(\partial\chi_1) = 0, \qquad \partial(\partial\chi_2) = 0 \tag{4.2}$$

Hence it follows that

$$\partial\chi_1 = C_1, \qquad \partial\chi_2 = C_2 \tag{4.3}$$

By formulas (2.6) we obtain the stresses corresponding to tension or compression by normal tractions distributed normally to the faces

$$\sigma_x = A_0, \qquad \sigma_y = \tau_{xy} = 0 \qquad (A_0 = -(s_1 + s_2)C_2) \tag{4.4}$$

B) *Antisymmetric distribution*. In the expansion of the operator Q^* the lowest power of ∂ will be the third. Discarding all powers of ∂ higher than the third, we obtain the equations

$$\partial^3(\partial\psi_1) = 0, \qquad \partial^3(\partial\psi_2) = 0 \tag{4.5}$$

Hence it follows that

$$\partial\psi_1 = C_1x^2 + D_1x + E_1, \qquad \partial\psi_2 = C_2x^2 + D_2x + E_2 \tag{4.6}$$

To these formulas correspond stresses connected with a bending moment and a shearing force

$$\sigma_x = (A_1x + B_1)y, \qquad \sigma_y = 0, \qquad \tau_{xy} = \frac{A_1}{2}\left(\frac{h^2}{4} - y^2\right)$$
$$(A_1 = -2s_1s_2(s_1 + s_2)C_1, \; B_1 = -s_1s_2(s_1 + s_2)(D_1 + hC_2)) \tag{4.7}$$

The solution of the problem for a strip of finite length is obtained by adding the homogeneous solutions (4.4) and (4.7) to that for the infinite strip. The three constants A_0, A_1, B_1, can always be selected so that the necessary conditions (integrated or averaged) on the end faces can be satisfied.

The method of solution for a strip will be illustrated with two examples.

5. Polynomial distribution of load. Equations (2.4) and (3.4) in expanded form can be written out in the following manner

$$(\partial + \alpha_3\partial^3 + \alpha_5\partial^5 + \ldots)\,\partial\chi_k = \frac{2p_k}{(s_1+s_2)h} \tag{5.1}$$

$$(\beta_3\partial^3 + \beta_5\partial^5 + \ldots)\partial\psi_k = \frac{2q_k}{(s_1+s_2)\,h} \tag{5.2}$$

where

$$\begin{array}{ll} p_1 = p, & \alpha_{2i+1} = (-1)^i \dfrac{(s_1+s_2)^{2i} + (s_1-s_2)^{2i}}{2^{2i+1}(2i+1)!}\,h^{2i} \\ p_2 = \tau, & \\ q_1 = q, & \qquad\qquad\qquad\qquad\qquad (i = 1,2,\ldots) \\ q_2 = t, & \beta_{2i+1} = (-1)^{i-1} \dfrac{(s_1+s_2)^{2i} - (s_1-s_2)^{2i}}{2^{2i+1}(2i+1)!}\,h^{2i} \end{array} \tag{5.3}$$

If p_k and q_k are given in the form of integral polynomials of x of degree n (where n is a positive integer), then it is easy to see that $\partial\chi_k$ will be a polynomial of degree $n+1$, and $\partial\psi_k$ will be a polynomial of degree $n+3$. The determination of the unknown coefficients of the functions $\partial\chi_k$, $\partial\psi_k$ does not present any great difficulty; we obtain the equation for them by equating coefficients of like powers of x on the left-hand and right-hand sides of (5.1) and (5.2).

Suppose, e.g. that one face of a strip is clamped while the other is free (a cantilever) and that the long edges are subjected to normal tractions distributed according to a cubic paraboloidal law $q_3x^3 : l^3$ (Fig. 3). The stresses can be built up from those resulting from the symmetric loading

$$p = -\frac{q_3}{2}\left(\frac{x}{l}\right)^3, \qquad \tau = 0 \tag{5.4}$$

and those from the antisymmetric loading

$$q = \frac{q_3}{2}\left(\frac{x}{l}\right)^3, \qquad t = 0 \tag{5.5}$$

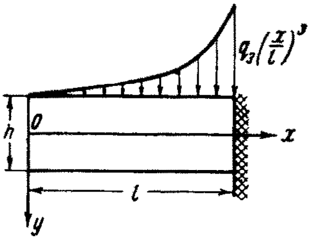

Fig. 3.

The first can be found with the aid of the functions

$$\partial\chi_1 = \frac{q_3}{4(s_1+s_2)hl^3}(-x^4+12\alpha_3 x^2), \quad \partial\chi_2 = 0 \tag{5.6}$$

Substituting these values into the formulas (2.6), it will be necessary henceforth to expand the sines and cosines into series and, after multiplication, to reject all powers of ∂ higher than the fourth (thus $\partial\chi_1$ will be a polynomial of fourth degree). We obtain

$$\begin{aligned}
\sigma_x &= k^2(s_1+s_2)\frac{h}{4}\left(\frac{h^2}{12}-y^2\right)\partial^3(\partial\chi_1) \\
\sigma_y &= 0.5(s_1+s_2)\left[h\partial(\partial\chi_1)-\frac{mh^3}{24}\partial^3(\partial\chi_1)\right] \\
\tau_{xy} &= -k^2(s_1+s_2)\frac{h^2}{12}\left(\frac{h^2 y}{4}-y^3\right)\partial^4(\partial\chi_1)
\end{aligned}
\qquad
\begin{aligned}
&\left(k = s_1 s_2 = \sqrt{\frac{\beta_{22}}{\beta_{11}}}\right) \\
&\left(m = s_1^2+s_2^2 = \frac{2\beta_{12}+\beta_{66}}{\beta_{11}}\right)
\end{aligned} \tag{5.7}$$

The final formulas for the stresses resulting from the symmetric loading will have the form

$$\sigma_x = \frac{3q_3}{2l^3}k^2\left(y^2-\frac{h^2}{12}\right)x, \qquad \sigma_y = -\frac{q_3x^3}{2l^3}, \quad \tau_{xy} = \frac{q_3k^2}{2l^3}\left(\frac{h^2y}{4}-y^3\right) \tag{5.8}$$

On the free and $x = 0$ the stress σ_x vanishes and τ_{xy} constitutes a self-equilibrating system of shearing forces so that the necessary conditions turn out to be satisfied.

For the functions determining the stresses due to the antisymmetric loading (5.5), we find the expressions

$$\partial\psi_1 = \frac{q_3}{120(s_1+s_2)hl^3\beta_3}\left(x^6-30\frac{\beta_5}{\beta_3}x^4\right) = \frac{q_3}{10(s_1+s_2)kh^3l^3}(x^6+0.75mh^2x^4), \ \partial\psi_2 = 0 \tag{5.9}$$

We now quote the final formulas for the stresses obtained from (3.6)

$$\begin{aligned}
\sigma_x &= \frac{3q_3}{2h^3l^3}\Big\{-\frac{2}{5}yx^5+\frac{m}{15}(20y^3-3h^2y)x^3+ \\
&\qquad +\frac{1}{40}[5k^2h^4y+8(m^2-5k^2)h^2y^3+16(m^2-k^2)y^5]\,x\Big\} \\
\sigma_y &= \frac{3q_3}{2h^3l^3}\left[\left(h^2y-\frac{4}{3}y^3\right)x^3+\frac{m}{40}(h^4y-8h^2y^3+16y^5)\,x\right] \\
\tau_{xy} &= \frac{3q_3}{2h^3l^3}\Big\{\left(y^2-\frac{h^2}{4}\right)x^4-\frac{m}{80}(h^4-24h^2y^2+80y^4)x^2+ \\
&\qquad +\frac{1}{960}[(2m^2+k^2)h^6-60k^2h^4y^2+48(5k^2-m^2)h^2y^4+64(m^2-k^2)y^6]\Big\}
\end{aligned} \tag{5.10}$$

On the free end we have $\sigma_x = 0$ and the shearing stress can be reduced to a force the magnitude of which, per unit thickness of the strip, is equal to

$$P = \frac{q_3 h^4}{11200 l^3}(27m^2 - 20k^2) \tag{5.11}$$

In order to eliminate this "superfluous" force we must add a distribution of stresses of the form (4.7)

$$\sigma_x = \frac{12P}{h^3}xy, \qquad \sigma_y = 0, \qquad \tau_{xy} = \frac{6P}{h^2}\left(\frac{h^2}{4} - y^2\right) \tag{5.12}$$

The complete stresses in the cantilever are obtained by superposition of the stresses determined by formulas (5.8), (5.10) and (5.12).

The expressions for the displacements will not be quoted. We merely mention that the arbitrary constants ω, u_0, v_0 from formulas (1.12) can be found from the additional conditions at the clamped end

$$u = v = \frac{\partial v}{\partial x} = 0 \quad \text{for } x = l, \ y = 0 \tag{5.13}$$

6. Loading given in the form of a Fourier series. Let the orthotropic strip of length $2l$ be loaded on the long edges by tractions represented in the form of Fourier series. An arbitrary problem of this type can be solved with the aid of the Airy stress function in the form of a series with the addition of polynomial terms (see, e.g. [1, pp.73-75]). In particular, the solution of the problem of a strip with free end faces, subjected to symmetric tractions was given in [5].

We will show that it is easy to derive the solution of this problem with the aid of the above-described, general method. It is sufficient to consider the particular case when the ends are free, and when the tractions are normal, symmetric and distributed on both edges $y = \pm h/2$ according to a cosine law, i.e.

$$p = p_n \cos\frac{n\pi x}{l}, \qquad \tau = q = t = 0 \tag{6.1}$$

where n is an arbitrary, real non-zero number and the origin of coordinates is placed at an arbitrary point on the axis of the strip.

In this case $\chi_2 = 0$, and χ_1 can be determined from the equation

$$Q(\partial\chi_1) = p_n \cos\frac{n\pi x}{l} \tag{6.2}$$

We point out as a preliminary some formulas which facilitate the finding of the solution of equation (6.2) and of the associated stresses and displacements

$$\sin\alpha\partial \times e^{ax} = \sin\alpha a\, e^{ax}, \qquad \cos\alpha\partial\times e^{ax} = \cos\alpha a\, e^{ax} \tag{6.3}$$

$$\sin\alpha\partial\cos\beta\partial \times \sin ax = \sinh\alpha a\cosh\beta a\cos ax$$
$$\cos\alpha\partial\cos\beta\partial \times \sin ax = \cosh\alpha a\cosh\beta a\sin ax$$
$$\sin\alpha\partial\sin\beta\partial \times \sin ax = -\sinh\alpha a\sinh\beta a\sin ax$$
$$\sin\alpha\partial\cos\beta\partial \times \cos ax = -\sinh\alpha a\cosh\beta a\sin ax$$
$$\cos\alpha\partial\cos\beta\partial \times \cos ax = \cosh\alpha a\cosh\beta a\cos ax$$
$$\sin\alpha\partial\sin\beta\partial \times \cos ax = -\sinh\alpha a\sinh\beta a\cos ax$$

By α, β and a is always meant an arbitrary, constant number not equal to zero, real or complex, or a linear function of y. The derivation of these formulas is elementary: the first two are found by expanding the sine- and cosine-operators into series of powers of ∂; the remainder follow by replacing the arbitrary trigonometric functions by sums or differences and finally expanding the operators in series.

Making use of (6.3), we find in the case $s_1 \neq s_2$

$$\partial\chi_1 = Ap_n(s_1 - s_2)\sin\frac{n\pi x}{l} \tag{6.4}$$

where

$$A = \frac{1}{s_1\sinh s_1\gamma\cosh s_2\gamma - s_2\sinh s_2\gamma\cosh s_1\gamma}, \qquad \gamma = \frac{n\pi h}{2l} \tag{6.5}$$

Substituting into (2.6) the value $\partial\chi_2 = 0$ and the expression for $\partial\chi_1$, and taking into account (6.3), we obtain the final formulas for the stresses

$$\begin{aligned}
\sigma_x &= Akp_n\left(s_1\sinh s_2\gamma\cosh\frac{s_1 n\pi y}{l} - s_2\sinh s_1\gamma\cosh\frac{s_2 n\pi y}{l}\right)\cos\frac{n\pi x}{l}\\
\sigma_y &= -Ap_n\left(s_2\sinh s_2\gamma\cosh\frac{s_1 n\pi y}{l} - s_1\sinh s_1\gamma\cosh\frac{s_2 n\pi y}{l}\right)\cos\frac{n\pi x}{l}\\
\tau_{xy} &= Akp_n\left(\sinh s_2\gamma\sinh\frac{s_1 n\pi y}{l} - \sinh s_1\gamma\sinh\frac{s_2 n\pi y}{l}\right)\sin\frac{n\pi x}{l}
\end{aligned} \tag{6.6}$$

On the ends of the strip, as well as in an arbitrary cross-section, the stresses constitute a self-equilibrating system of forces (the resultant force and moment vanish), and consequently the required conditions on the free faces have been satisfied (as is sometimes said, "with an accuracy according to Saint-Venant's principle").

It is just as simple to find the stresses also in the cases when the tractions are distributed according to an exponential or hyperbolic law.

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